

# A Quantum Particle Undergoing Continuous Observation

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December 1988

Published in: *Physics Letters A*, **140** (1989) No 7,8, pp 359  
–362

## Abstract

A stochastic model for the continuous nondemolition observation of the position of a quantum particle in a potential field and a boson reservoir is given. It is shown that any Gaussian wave function evolving according to the posterior wave equation with a quadratic potential collapses to a Gaussian wave packet given by the stationary solution of this equation..

The recently developed methods of quantum stochastic calculus [7, ?] can serve for the description of the time-development of continuously observed quantum systems [1, 4, 5].

We apply this approach to describe the time-behaviour of a one-dimensional quantum particle in the field of the linear force  $F = kx + mg$ . The effect of coupling of the particle to a measuring apparatus is represented by an extra stochastic force. Assuming a non-ideal indirect observation of the particle position one can select such an observation channel that the observation is nondemolition [4, 5], i.e. it does not affect the actual as well as the future states of the perturbed particle. The posterior dynamics of the observed particle is then given by the nonlinear stochastic wave equations rigorously derived by quantum filtering method in [4, 3].

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It is shown that the Gaussian wave packet evolves to the asymptotic stationary one the width of which (identified with the standard deviation) in the coordinate representation is given by the formula  $\tau_q = \hbar/2m[(\kappa^2 + \lambda^2)^{1/2} - \kappa]^{1/4}$ , where  $\kappa = k/\hbar$  and  $\lambda$  is the accuracy coefficient of the nondemolition measurement of the particle position. This phenomenon which cannot be explained by the Schrödinger equation (describing the time-evolution of the unobserved quantum system) belongs to the class of phenomena called watchdog effects [?].

Let us assume that the measuring apparatus is modelled by the Bose field. The motion of the particle in a potential  $\phi$  is distorted by the apparatus so the time-derivative  $\dot{P}$  of the momentum of the particle is no longer equal to  $F(X) = -\phi'(X)$  ( $\phi' = \partial\phi/\partial x$ ). The Heisenberg equations describing the time-development of the momentum and the position of the observed particle have the form [6]

$$\dot{P}(t) - F(X(t)) = f(t), \quad \dot{X}(t) = \frac{1}{m}P(t), \quad (1)$$

where the force  $f(t)$  in our model is taken as

$$f(t) = \frac{\hbar}{i}(\lambda/2)^{1/2}[a^\dagger(t) - a(t)] = (2\lambda)^{1/2}\hbar\Im a^\dagger(t), \quad \lambda > 0. \quad (2)$$

Here  $a(t) = b_0(t)$ ,  $a^\dagger(t) = b_0^\dagger(t)$  are the annihilation and creation quantum noise operators with the canonical commutation relations

$$[a(t), a^\dagger(t)] = \delta(t - t'), \quad [a(t), a(t')] = 0. \quad (3)$$

given by the standard boson field operators  $b_s, b_s^\dagger$ , on the half of line  $s \leq 0$  with free evolution  $b_s(t) = b_{s-ct}$ ,  $b_s^\dagger = b_{s-ct}^\dagger$  at  $s = 0$ .

The position of the particle is assumed to be observed indirectly, together with some noise  $e(t)$  (error), therefore the measured quantity is

$$y(t) = (2\lambda)^{1/2}X(t) + e(t). \quad (4)$$

it is easy to check with the help of (3) that if

$$e(t) = a(t) + a^\dagger(t) = 2\Re a(t) \quad (5)$$

then for  $P(t)$  and  $X(t)$  satisfying (1) the following commutation relations hold,

$$[P(t), y(t')] = 0, \quad [X(t), y(t')] = 0, \quad \forall t' \leq t. \quad (6)$$

From these relations it follows that the preparation for the measurement of any functional  $Y$  of the past operators  $y(t')$ ,  $t' \leq t$  does not affect  $P(t)$  and  $X(t)$  as well as any other Heisenberg operator  $Z(t')$  of the particle at  $t' \geq t$ . In other words, the condition (6) means that the measurement of  $Y$  disturbs *a priori* neither the present nor the future state of the particle.

Note, that  $y(t) = y^\dagger(t)$  and

$$[y(t), y(t')] = 0, \quad \forall t, t', \quad (7)$$

therefore  $y(t)$  can be continuously measured in time as a classical quantity. Following refs. [4, 2], one can say that the continual measurement of  $y$  is nondemolition with respect to the time-evolution of the system.

Obviously, the minimal distortion of motion (1) will be obtained for the Bose field in a vacuum state. In such a situation  $e$  and  $f$  are white noises,  $e$  has standard intensity 1, while the intensity of  $f$  is proportional to the measurement accuracy:

$$\begin{aligned} \langle f(t) \rangle &= \langle e(t) \rangle = 0, & \langle f(t)f(t') \rangle &= \frac{1}{2} \lambda \hbar^2 \delta(t - t'), \\ \langle e(t)e(t') \rangle &= \delta(t - t'). \end{aligned} \quad (8)$$

It is convenient to rewrite the equations of motion (1) in the form of the Ito quantum stochastic differential equations [7, 4, 5, 6]. We obtain

$$\begin{aligned} dP(t) &= \frac{i}{\hbar} [\phi(X(t)), P(t)] dt + (\lambda/2)^{1/2} [X(t), P(t)] [dA(t) - dA^\dagger(t)], \\ dX(t) &= \frac{i}{\hbar} [P^2(t)/2m, X(t)] dt, \end{aligned} \quad (9)$$

where  $dA(t) = A(t + dt) - A(t)$  is the stochastic differential of the standard quantum Brownian motion,  $A(t)|_{t=0} = 0$ , with the generalized derivative  $a(t) = dA(t)/dt$ . Using the quantum Ito formula [7] one can obtain from eqs. (9) the time-evolution equation for any observable  $Z$  (a self-adjoint polynomial in  $X$  and  $P$ )

$$\begin{aligned} dZ(t) &= \{ -(i/\hbar) [Z(t), P^2(t)/2m + \phi(X(t))] - \frac{1}{4} \lambda [X(t), [X(t), Z(t)]] \} dt \\ &+ 2\Re \{ (\lambda/2)^{1/2} [X(t), Z(t)] dA \}. \end{aligned} \quad (10)$$

Eq. (10) describes the prior stochastic dynamics of the particle considered as an open quantum system; the term “prior dynamics” means that the process  $Z(t)$  is not conditioned by the results of the observation. Obviously, the prior state of the particle in the Schrödinger picture is described by a mixed density matrix even if the initial state is pure (given by the particle wave function  $\Psi$  and the Fock vacuum vector).

The Bose field does not only disturb the system but also conveys some information about it. This information is contained in the “output field”  $a(t)_{\text{out}}$  — the field after interaction with the system in question. The measured quantity  $y(t)$  appearing in (4) can be interpreted as  $2\Re a(t)_{\text{out}}$  while  $e(t) = 2\Re a(t)$  with  $a(t)$  being the input field [6]. Let us rewrite (4) in the form of Ito quantum stochastic differential

$$dY(t) = (2\lambda)^{1/2} X(t)dt + 2\Re dA(t). \quad (11)$$

for the integral  $Y(t) = \int_0^t y(t')dt'$ .

The posterior mean values  $\hat{z}(t)$ , i.e. the mean values of the process  $Z(t)$  which is partially observed by means of  $Y(t)$ , are defined as conditional expectations  $\hat{z}(t) = \epsilon^t(Z(t))$  with respect to the observables  $Y^t = \{Y(s)|s \leq t$ . For a given  $Z$ ,  $\hat{z}$  is a non-anticipating functional of the observed trajectories  $q = \{q(t)\}$  of the output process  $Y$ . If the initial state of the system is pure and the initial state of the bath is a vacuum state then the posterior expectation values are realized with the help of the stochastic wave function called the posterior wave function. The posterior state is therefore a pure one [5]. According to refs. [4, 5], the posterior stochastic wave function  $\hat{\varphi}(t, x)$  satisfies a new nonlinear stochastic (posterior) wave equation which in our case has the form

$$d\hat{\varphi} + \left[ \frac{i\hbar}{2m} \hat{\varphi}'' + \left( \frac{i}{\hbar} \phi + \frac{\lambda}{4} (x - \hat{q})^2 \right) \hat{\varphi} \right] dt = (\lambda/2)^{1/2} (x - \hat{q}) x \hat{\varphi} d\tilde{Y}, \quad (12)$$

with the initial condition  $\hat{\varphi}(0, x) = \psi(x)$ . In the above formula

$$\hat{q}(t) = \int \hat{\varphi}^*(t, x) x \hat{\varphi}(t, x) dx \quad (13)$$

and

$$d\tilde{Y}(t) = dY(t) - (2\lambda)^{1/2} \hat{q}(t)dt \quad (14)$$

denotes the ito differential of the observed commutative innovating process, which is equivalent to the standard Wiener one as  $\langle d\tilde{Y}(t) \rangle = 0$  and  $(d\tilde{Y}(t))^2 = dt$ .

Let us now discuss the problem of the time-development of the posterior wave function at  $t \rightarrow \infty$ . We shall assume that the initial state is the Gaussian wave packet,

$$\psi(x) = (2\pi\sigma_q^2)^{1/4} \exp\left(-\frac{1}{4\sigma_q^2}(x-q)^2 + \frac{i}{\hbar}px\right), \quad (15)$$

where  $p$  and  $q$  denote initial mean values of position and momentum of the particle and  $\sigma_q^2$  stands for the initial dispersion of the wave packet in the coordinate representation. We shall prove that the solution  $\hat{\varphi}(t, x)$  of eq. (12) corresponding to the initial condition (15) has the form of a Gaussian packet,

$$\hat{\varphi}(t, x) = c(t) \exp\{(1/\hbar)[m\omega(t)[x - \hat{q}(t)]^2/2 + i\hat{p}(t)x]\}, \quad (16)$$

with the posterior mean values  $\hat{q}(t)$  (given by (13)) and

$$\hat{p}(t) = \frac{\hbar}{i} \int \hat{\varphi}^*(t, x) \hat{\varphi}'(t, x) dx \quad (17)$$

fulfilling linear filtration equations and  $\omega(t)$  satisfying the Riccati differential equation. The normalization factor  $c(t) = (2\pi\tau_q^2)^{-1/4}$  up to an inessential phase multiplier and  $\tau_q^2 = \widehat{q^2} - \hat{q}^2$  is a posterior dispersion of position.

For this purpose it is convenient to rewrite eq. (12) in terms of complex osmotic velocity. Let us first introduce

$$T(t, x) = R(t, x) + iS(t, x) = \hbar \ln \hat{\varphi}(t, x). \quad (18)$$

Then by Ito's rule

$$dG(\hat{\varphi}) = G'(\hat{\varphi})d\hat{\varphi} + \frac{1}{2}G''(\hat{\varphi})(d\hat{\varphi})^2$$

applied to the function  $G = \hbar \ln \hat{\varphi}$  and taking into account that  $(d\hat{\varphi})^2 = \frac{1}{2}\lambda(x - \hat{q})^2\hat{\varphi}^2dt$  we obtain eq. (12) in the form

$$dT + \left[\frac{1}{2}\hbar\lambda(x - \hat{q})^2 + i\phi - (i/2m)(T'^2 + \hbar T'')\right]dt = (\lambda/2)^{1/2}\hbar(x - \hat{q})d\tilde{Y}. \quad (19)$$

In terms of complex osmotic velocity

$$W(t, x) = \frac{1}{m} T'(t, x) = U(t, x) + iV(t, x) \quad (20)$$

eq. (19) can be rewritten as

$$dW + [(\hbar\lambda/m)(x - \hat{q}) + (i/\hbar)\phi' - i(WW' + \hbar W''/2m)]dt = (\lambda/2)^{1/2} \frac{\hbar}{m} d\tilde{Y}. \quad (21)$$

We look for the solution of (21) corresponding to the linear force  $F(x) = \hbar\kappa x + mg$  and to the initial condition

$$W(0, x) = \frac{\hbar}{m} \frac{\Psi'(x)}{\Psi(x)} = \frac{\hbar}{2m\sigma_q^2} (x - q) + \frac{i}{m} p \quad (22)$$

in the linear form

$$W(t, x) = \hat{w}(t) + \omega(t)x, \quad (23)$$

where in accordance with (16)

$$\hat{w} = -\omega\hat{q} + \frac{i}{m} \hat{p}. \quad (24)$$

By inserting  $W' = \omega$ ,  $W'' = 0$  into (21) we obtain the following system of equations for the coefficients  $\hat{w}(t) = W(t, 0)$  and  $\omega(t) = W'(t, 0)$ ,

$$d\hat{w}(t) - i[g + \omega(t)\hat{w}(t)]dt = (\lambda/2)^{1/2} \frac{\lambda}{m} d\tilde{Y}(t), \quad (25)$$

with  $\hat{w}(0) = \frac{\hbar}{2m\sigma_q^2} q + \frac{i}{\hbar} p$ ,

$$\frac{d}{dt} \omega(t) + \frac{\hbar\lambda}{m} = i[\hbar\kappa/m + \omega^2(t)], \quad (26)$$

with  $\omega(0) = -\frac{\hbar}{2m\sigma_q^2}$ , which define the solution of (21) in the form given by (23). From (24) we get  $\hat{q}(t) = -\Re\hat{w}(t)/2\Re\omega(t)$  which is the root of the equation  $R'(t, x) = mU(t, x) = 0$  for which the maximum of the posterior density  $|\hat{\varphi}(t, x)|^2 = \exp[(2/\hbar)R(t, x)]$  is attained. The posterior momentum  $\hat{p}$  coincides with  $mV(t, \hat{q}) = S'(t, x)_{x=\hat{q}}$  and by (24)  $\hat{p}(t) = m\Im[\hat{w}(t) + \omega(t)\hat{q}(t)]$ .

Eqs. (24)–(26) give the Hamilton-Langevin equations describing the time-development of posterior mean values of position and momentum,

$$\begin{aligned}\hat{p}dt - m d\hat{q} &= \hbar(\lambda/2)^{1/2} \frac{d\tilde{Y}}{\Re \omega}, & \hat{q}(0) &= q, \\ d\hat{p} - mgdt &= \hbar \left( \kappa \hat{q}dt - (\lambda/2)^{1/2} \frac{\Im \omega}{\Re \omega} d\tilde{Y} \right), & \hat{p}(0) &= p.\end{aligned}\tag{27}$$

The posterior position and momentum dispersions for the posterior wave function in the form (16) are given by the formulas

$$\tau_q^2(t) = -\frac{\hbar}{2m\Re \omega(t)}, \quad \tau_p^2(t) = -\frac{\hbar m |\omega(t)|^2}{2\Re \omega(t)},\tag{28}$$

with  $\omega(t)$  being the solution of eq. (26); therefore the Heisenberg inequality  $\tau_q^2(t)\tau_p^2(t) \geq \hbar^2/4$  is fulfilled.

The general solution of eq. (26) reads

$$\omega(t) = i\alpha \frac{\omega(0) + i\alpha \tanh(\alpha t)}{i\alpha + \omega(0) \tanh(\alpha t)}, \quad \alpha = (\hbar/m)^{1/2}(\kappa + i\lambda)^{1/2}.\tag{29}$$

Obviously,  $\lim_{t \rightarrow \infty} \omega(t) = i\alpha$ , i.e.  $i\alpha$  is the asymptotic stationary solution of (26). Consequently, posterior dispersions of position and momentum tend to finite limits independent of their initial values, from (28) and (29) we get

$$\begin{aligned}\tau_q^2(\infty) &= \frac{\hbar}{2m\Im \alpha} = \left( \frac{\hbar}{2m} \right)^{1/2} [(\kappa^2 + \lambda^2)^{1/2} - \kappa]^{-1/2}, \\ \tau_p^2(\infty) &= \frac{\hbar m |\alpha|^2}{2\Im \alpha} = \left( \frac{\hbar^3 m}{2} \right)^{1/2} \left( \frac{\kappa^2 + \lambda^2}{(\kappa^2 + \lambda^2)^{1/2} - \kappa} \right)^{1/2}.\end{aligned}\tag{30}$$

Let us pay attention to the particular watchdog effects which can be obtained from (30).

- (a) For  $\kappa < 0$ , i.e. for the harmonic oscillator we find that the width of the stationary Gaussian packet is smaller than that for the unobserved oscillator. If the accuracy  $\lambda \rightarrow \infty$  then  $\tau_q^2(\infty) \rightarrow 0$  but  $\tau_p^2(\infty) \rightarrow \infty$ .
- (b) For  $\kappa = 0$ , i.e. for a particle in a homogeneous (gravitation or electric) field, in particular for a free observed particle ( $g = 0$ ) the Gaussian packet does not spread out in time. The asymptotic localization  $\tau_q^2(\infty) = (\hbar/2m\lambda)^{1/2}$  is inversely proportional to the mass of the particle and the measurement accuracy coefficient.

- (c) For  $\kappa > 0$ , i.e. for the case of a linear active system (harmonic accelerator) we obtain a similar watchdog effect as in case (b).

Thus the collapse problem of the wave function for a quantum particle under the position measurement has found the dynamical solution.

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